COMMENTS ON "SEMI-GRAPHS OF ANABELIOIDS"

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(1.) In the third sentence of Example 2.10, the phrase "Now observe that a hyperbolic Riemann surface of finite type" should read *"Now observe that a hyperbolic Riemann surface of finite type of genus* ≥ 1*"*.

(2.) With regard to the proof of Corollary 3.11:

(i) In the first line of the proof, it should be stipulated that the set Σ be *nonempty*.

(ii) The phrase *"as in (ii)"* in line 2 of observation (iv) should read *"as in (iii)"*.

(iii) A more detailed version of the argument used to verify observation (iv) is given in [AbsTopII], Corollary 2.11.

(3.) In the discussion of the *"pro-*Σ *version" of Corollary 3.11* in Remark 3.11.1,

one should *assume* that $p_\alpha, p_\beta \in \Sigma$.

In fact, this assumption is, in some sense, *implicit* in the phraseology that appears in the first two lines of Remark 3.11.1, but it should have been stated *explicitly*.

(4.) Note that in Theorem 5.4, the case where $\mathcal{A}$ is *trivial* [i.e., is equal to the anabelioid associated to the *trivial group* $\{1\}$] is *not excluded*. Thus, suppose that, in Theorem 5.4, we assume further that $\mathcal{A}$ is *trivial*. Then let us *observe* that this implies that the underlying graph of $\overline{\mathcal{G}}$ [or $\overline{\mathcal{H}}$] consists of a *single vertex* and *no edges*. [Indeed, if the underlying graph of $\overline{\mathcal{G}}$ has at least one edge, then since $\overline{\mathcal{G}}$ is assumed to be *totally elevated*, it follows from the assumption that $\overline{\mathcal{G}}$ is *totally arithmetically estranged* [cf. Definition 5.3, (ii)] that $\Pi^{\mathrm{temp}}_{\overline{\mathcal{G}}}$ admits a closed subgroup that *fails to be arithmetically ample*, hence that $\Pi_{\mathcal{A}} = \{1\}$ contains a closed subgroup which is *not open* — a *contradiction*.] Thus, $\Pi^{\mathrm{temp}}_{\overline{\mathcal{G}}}$ itself is a *verticial subgroup* of $\Pi^{\mathrm{temp}}_{\overline{\mathcal{G}}}$, hence *compact*. In particular, $\Pi^{\mathrm{temp}}_{\overline{\mathcal{G}}}$ is the *unique maximal compact subgroup* of $\Pi^{\mathrm{temp}}_{\overline{\mathcal{G}}}$, so assertions (i), (ii), and (iii) of Theorem 5.4 are, in essence, *vacuous*.

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(5.) In the line 2 of Example 5.6, the phrase "Also, Suppose..." should read "Also, suppose...". In lines 7–8 of Example 5.6, one should also assume that M_i was chosen so that the resulting Galois action on the dual semi-graph with compact structure of the special fiber of the stable model is *trivial* [i.e., so as to ensure that the assumption of Theorem 5.4 concerning switching the branches of edges is satisfied].

(6.) Some readers may find the argument given in the third and fourth paragraphs of the proof of Theorem 3.7, (iii), to be a bit confusing in its brevity. A more detailed argument may be given as follows. For $i \in I$, let us write

$$\mathbb{V}_i, \ \mathbb{E}_i$$

for the sets of *vertices* and *closed edges*, respectively, of $\mathbb{G}_{i,\infty}$ that are *fixed* by the action of H. Thus, for $i \geq j \in I$, we have natural maps $\mathbb{V}_i \to \mathbb{V}_j$, $\mathbb{E}_i \to \mathbb{E}_j$; let us write

$$\mathbb{E}_{j,i} \subseteq \mathbb{E}_j$$

for the image of $\mathbb{E}_i$ in $\mathbb{E}_j$. Thus, for $i_1, i_2 \in I$ such that $i_1 \geq i_2$, we have $\mathbb{E}_{j,i_1} \subseteq \mathbb{E}_{j,i_2} \subseteq \mathbb{E}_j$. Also, we recall that, by the argument given in the second paragraph of the proof, we have $\#\mathbb{V}_i \geq 1$ [where we use the notation "#" to denote the cardinality of a set], for all $i \in I$. For simplicity, in the following, we assume that the semi-graph $\mathbb{G}_i$ is *untangled*, for all $i \in I$. Now:

(a) Suppose that for some cofinal subset $J \subseteq I$, we have $\#\mathbb{V}_j = 1$, for all $j \in J$. Then the unique elements of the $\mathbb{V}_j$, for $j \in J$, form a *compatible system of vertices fixed by* H. Thus, we conclude that H is contained in some *verticial subgroup* of $\pi_1^{\text{temp}}(\mathcal{G})$.

(b) Suppose that for some cofinal subset $J \subseteq I$, we have $\#\mathbb{V}_j \geq 2$, for all $j \in J$. Then it follows from Lemma 1.8, (ii), (b), that $\#\mathbb{E}_j \geq 1$, for all $j \in J$. Now I *claim* that for each $j \in J$, the following condition holds:

$(*_j)$ there exists an $i \in J$ such that $i \geq j$ and $\#\mathbb{E}_{j,i} = 1$.

Indeed, suppose that $(*_j)$ *fails to hold*. Then for each $i \geq j$ in J, there exists a pair of *distinct edges* $e_i, e'_i \in \mathbb{E}_i$ whose respective images $e_{j,i}, e'_{j,i} \in \mathbb{E}_j$ are *distinct*. By Lemma 1.8, (ii), (b), we may assume without loss of the generality that the pair $\{e_i, e'_i\}$, hence also the pair $\{e_{j,i}, e'_{j,i}\}$, forms a *subjoint*. Then since $\mathbb{G}_j$ is *untangled*, it follows that the respective images $f_{j,i}$, $f'_{j,i}$ of $e_{j,i}$, $e'_{j,i}$ in $\mathbb{G}_j$ also form a *subjoint*. Write f_i, f'_i for the respective images of e_i, e'_i in $\mathbb{G}_i$. Thus, it follows from the fact that the pair $(f_{j,i}, f'_{j,i})$ forms a subjoint (of $\mathbb{G}_j$) that the pair (f_i, f'_i) forms a subjoint (of $\mathbb{G}_i$). Moreover, for some cofinal subset $J^* \subseteq J$, the subjoints (f_i, f'_i), where $i \in J^*$, converge, in the profinite topology, to some *profinite subjoint*. As discussed in the third paragraph, this leads to a *contradiction*, in light of our assumption that $\mathcal{G}$ is *totally estranged*. This completes the proof of the *claim*. Now it follows from $(*_j)$ that each of the *nonempty* sets $\mathbb{E}_{j,i}$, for $i, j \in J$ such that i is *“sufficiently large”* relative to j, is of cardinality 1. But this implies that each intersection

$$\mathbb{E}_{j,\infty} \stackrel{\mathrm{def}}{=} \bigcap_{i \geq j} \mathbb{E}_{j,i}$$

is of cardinality 1. Thus, the unique elements of the $\mathbb{E}_{j,\infty}$, for $j \in J$, form a *compatible system of closed edges fixed by H*. In particular, we conclude that H is contained in some *edge-like subgroup*, hence also in *two distinct verticial subgroups*, of $\pi_1^{\mathrm{temp}}(\mathcal{G})$.

(c) Now it follows formally from (a), (b) that H is *always* contained in *some verticial subgroup* of $\pi_1^{\mathrm{temp}}(\mathcal{G})$. If H is contained in *three distinct verticial subgroups*, then it follows immediately from Lemma 1.8, (ii), (b), that one obtains a *contradiction* to the condition $(*_j)$ of (b). This completes the proof of assertion (iii) of Theorem 3.7.

(7.) In Proposition 4.4, (ii), the notation “$\overline{\mathcal{G}}$” should read “$\mathcal{G}$”.

(8.) In the context of Theorem 4.8, it should be observed that $\mathcal{G}_i$ is assumed to be a *graph* [i.e., not an arbitrary semi-graph!] of anabelioids. Also, it should be observed that it follows immediately from the assumption that $\mathcal{G}_i$ is *totally aloof*, together with the definition of the category $\mathfrak{Loc}(\mathcal{G}_i, \Gamma_i)$, that the map induced on *branches* of underlying semi-graphs by a locally trivial morphism of $\mathfrak{Loc}(\mathcal{G}_i, \Gamma_i)$ is *completely determined* by the map induced [by the morphism under consideration] on *vertices* of underlying semi-graphs.

(9.) The *assertion* stated in the second display of Remark 2.4.2 is *false* as stated. [The automorphisms of the semi-graphs of anabelioids in Example 2.10 that arise from *“Dehn twists”* constitute a well-known counterexample to this *assertion*.] This *assertion* should be replaced by the following slightly modified version of this *assertion*:

> The isomorphism classes of the ϕ_v completely determine the isomorphism class of each of the ϕ_e, as well as each isomorphism ϕ_b, up to composition with an automorphism of the composite 1-morphism of anabelioids $\mathcal{G}_e \to \mathcal{H}_f \to \mathcal{H}_w$ that arises from an automorphism of the 1-morphism of anabelioids $\mathcal{G}_e \to \mathcal{H}_f$.

Also, in the *discussion* following this *assertion* [as well as the various places where this *discussion* is applied, i.e., Remark 3.5.2; the second paragraph of §4; Definition 5.1, (iv)], it is necessary to assume further that the semi-graphs of anabelioids that appear satisfy the condition that *every edge abuts to at least one vertex*.

(10.) The phrase “is *Galois*” at the end of the first sentence of the proof of Proposition 3.2 should read “is a countable coproduct of *Galois* objects”.

(11.) In the first sentence of Definition 3.5, (ii), the phrase "Suppose that" should read "Suppose that each connected component of"; the phrase "*splits* the restriction of" should read "*splits* the restriction of this connected component of".

(12.) Certain pathologies occur in the theory of tempered fundamental groups if one does not impose suitable *countability* hypotheses.

(i) In order to discuss these countability hypotheses, it will be convenient to introduce some *terminology* as follows:

(T1) We shall say that a tempered group is *Galois-countable* if its topology admits a countable basis. We shall say that a connected temperoid is *Galois-countable* if it arises from a Galois-countable tempered group. We shall say that a temperoid is *Galois-countable* if it arises from a collection of Galois-countable connected temperoids. We shall say that a connected quasi-temperoid is *Galois-countable* if it arises from a Galois-countable connected temperoid. We shall say that a quasi-temperoid is *Galois-countable* if it arises from a collection of Galois-countable connected quasi-temperoids.

(T2) We shall say that a semi-graph of anabelioids $\mathcal{G}$ is *Galois-countable* if it is countable, and, moreover, admits a countable collection of finite étale coverings $\{\mathcal{G}_i \to \mathcal{G}\}_{i \in I}$ such that for any finite étale covering $\mathcal{H} \to \mathcal{G}$, there exists an $i \in I$ such that the base-changed covering $\mathcal{H} \times_{\mathcal{G}} \mathcal{G}_i \to \mathcal{G}_i$ *splits* over the constituent anabelioid associated to each component of [the underlying semi-graph of] $\mathcal{G}_i$.

(T3) We shall say that a semi-graph of anabelioids $\mathcal{G}$ is *strictly coherent* if it is coherent [cf. Definition 2.3, (iii)], and, moreover, each of the profinite groups associated to components c of [the underlying semi-graph of] $\mathcal{G}$ [cf. the final sentence of Definition 2.3, (iii)] is *topologically generated by N generators*, for some positive integer N that is *independent* of c. In particular, it follows that if $\mathcal{G}$ is *finite* and *coherent*, then it is *strictly coherent*.

(T4) One verifies immediately that every *strictly coherent*, countable semi-graph of anabelioids is *Galois-countable*.

(T5) One verifies immediately that if, in Remark 3.2.1, one assumes in addition that the temperoid $\mathcal{X}$ is *Galois-countable*, then it follows that its associated *tempered fundamental group* $\pi_1^{\mathrm{temp}}(\mathcal{X})$ is *well-defined* and *Galois-countable*.

(T6) One verifies immediately that if, in the discussion of the paragraph preceding Proposition 3.6, one assumes in addition that the semi-graph of anabelioids $\mathcal{G}$ is *Galois-countable*, then it follows that its associated *tempered fundamental group* $\pi_1^{\mathrm{temp}}(\mathcal{G})$ and *temperoid* $\mathcal{B}^{\mathrm{temp}}(\mathcal{G})$ are *well-defined* and *Galois-countable*.

Here, we note that, in (T5) and (T6), the *Galois-countability* assumption is necessary in order to ensure that the index sets of *"universal covering pro-objects"* implicit in the definition of the *tempered fundamental group* may to be taken to be *countable*. This countability of the index sets involved implies that the various objects that constitute such a universal covering pro-object admit a *compatible system of basepoints*, i.e., that the *obstruction* to the existence of such a compatible system — which may be thought of as an element of a sort of "nonabelian $\mathbb{R}^1 \varprojlim$" — *vanishes*. In order to define the tempered fundamental group in an intrinsically meaningful fashion, it is necessary to know the existence of such a compatible system of basepoints.

(ii) The *effects* of the *omission* of *Galois-countability hypotheses* in §3 on the remainder of the present paper, as well as on subsequent papers of the author, may be summarized as follows:

(E1) First of all, we observe that all topological subquotients of *absolute Galois groups of fields of countable cardinality* are *Galois-countable*.

(E2) Also, we observe that if k is a field whose absolute Galois group is *Galois-countable*, and U is a nonempty open subscheme of a connected proper k-scheme X that arises as the underlying scheme of a log scheme that is log smooth over k [where we regard $\mathrm{Spec}(k)$ as equipped with the trivial log structure], and whose interior is equal to U, then the *tamely ramified arithmetic fundamental group* of U [i.e., that arises by considering finite étale coverings of U with *tame ramification* over the divisors that lie in the complement of U in X] is itself *Galois-countable* [cf., e.g., [AbsTopI], Proposition 2.2].

(E3) Next, we observe, with regard to Examples 2.10, 3.10, and 5.6, that the tempered groups and temperoids that appear in these Examples are *Galois-countable* [cf. (E1), (E2)], while the semi-graphs of anabelioids that appear in these Examples are *strictly coherent* [cf. item (T3) of (i)], hence [cf. item (T4) of (i)] *Galois-countable*. In particular, there is *no effect* on the theory of objects discussed in these Examples.

(E4) It follows immediately from (E3) that there is *no effect* on §6.

(E5) It follows immediately from items (T3), (T4) of (i), together with the assumptions of *finiteness* and *coherence* in the discussion of the paragraph immediately preceding Definition 4.2, the assumption of *coherence* in Definition 5.1, (i), and the assumption of Definition 5.1, (i), (d), that there is *no effect* on §4, §5. [Here, we note that since the notion of a *tempered covering* of a semi-graph of anabelioids is only defined in the case where the semi-graph of anabelioids is *countable*, it is implicit in Proposition 5.2 and Definition 5.3 that the semi-graphs of anabelioids under consideration are *countable*.]

(E6) There is *no effect* on §1, §2, or the Appendix, since *tempered fundamental groups* are *never discussed* in §1, §2, or the Appendix.

(E7) In the Definitions/Propositions/Theorems/Corollaries numbered 3.2, 3.3, 3.4, 3.5, 3.6, 3.7, 3.8, 3.9, one must assume that all tempered groups, temperoids, and semi-graphs of anabelioids that appear are *Galois-countable*.

On the other hand, it follows immediately from (E1), (E2), and (E3) that there is *no effect* on the remaining portions of §3.

(E8) In [QuCnf] and [FrdII], one must assume that all tempered groups and [quasi-]temperoids that appear are *Galois-countable*.

(E9) There is *no effect* on any papers of the author other than the present paper and the papers discussed in (E8).

(13.) In order to carry out the argument stated in the proof of Proposition 5.2, (i), it is necessary to *strengthen* the conditions (c) and (d) of Definition 5.1, (i), as follows. This strengthening of the conditions (c) and (d) of Definition 5.1, (i), has *no effect* either on the remainder of the present paper or on subsequent papers of the author. Suppose that $\mathcal{G}$ is as in Definition 5.1, (i). Then we begin by making the following *observation*:

(O1) Suppose that $\mathcal{G}$ is *finite*. Then $\mathcal{G}$ admits a *cofinal, countable* collection of *connected finite étale Galois coverings* $\{\mathcal{G}^i \to \mathcal{G}\}_{i \in I}$, each of which is *characteristic* [i.e., any pull-back of the covering via an element of $\mathrm{Aut}(\mathcal{G})$ is isomorphic to the original covering]. [For instance, one verifies immediately, by applying the *finiteness* and *coherence* of $\mathcal{G}$, that such a collection of coverings may be obtained by considering, for n a positive integer, the *composite* of all connected finite étale Galois coverings of degree $\leq n$.] We may assume, without loss of generality, that this collection of coverings arises from a *projective system*, which we denote by $\widetilde{\mathcal{G}}$. Thus, we obtain a *natural exact sequence*

$$1 \longrightarrow \mathrm{Gal}(\widetilde{\mathcal{G}}/\mathcal{G}) \longrightarrow \mathrm{Aut}(\widetilde{\mathcal{G}}/\mathcal{G}) \longrightarrow \mathrm{Aut}(\mathcal{G}) \longrightarrow 1$$

— where we write "$\mathrm{Aut}(\widetilde{\mathcal{G}}/\mathcal{G})$" for the group of pairs of *compatible* automorphisms of $\widetilde{\mathcal{G}}$ and $\mathcal{G}$.

This observation (O1) has the following immediate consequence:

(O2) Suppose that we are in the situation of (O1). Consider, for $i \in I$, the *finite index normal subgroup*

$$\mathrm{Aut}^i(\widetilde{\mathcal{G}}/\mathcal{G}) \quad \subseteq \quad \mathrm{Aut}(\widetilde{\mathcal{G}}/\mathcal{G})$$

of elements of $\mathrm{Aut}(\widetilde{\mathcal{G}}/\mathcal{G})$ that induce the *identity* automorphism on the underlying semi-graph $\mathbb{G}^i$ of $\mathcal{G}^i$, as well as on $\mathrm{Gal}(\mathcal{G}^i/\mathcal{G})$. Then one verifies immediately [from the definition of a *semi-graph of anabelioids*; cf. also Proposition 2.5, (i)] that the intersection of the $\mathrm{Aut}^i(\widetilde{\mathcal{G}}/\mathcal{G})$, for $i \in I$, is $= \{1\}$. Thus, the $\mathrm{Aut}^i(\widetilde{\mathcal{G}}/\mathcal{G})$, for $i \in I$, determine a *natural profinite topology* on $\mathrm{Aut}(\widetilde{\mathcal{G}}/\mathcal{G})$ and hence also on the quotient $\mathrm{Aut}(\mathcal{G})$, which is easily seen to be compatible with the profinite topology on $\mathrm{Gal}(\widetilde{\mathcal{G}}/\mathcal{G})$ and, moreover, *independent* of the choice of $\widetilde{\mathcal{G}}$.

The *new version* of the condition (c) of Definition 5.1, (i), that we wish to consider is the following:

(c^{new}) The action of H on $\mathbb{G}$ is trivial; the resulting homomorphism $H \to \mathrm{Aut}(\mathcal{G}[c])$, where c ranges over the *components* [i.e., vertices and edges] of $\mathbb{G}$, is *continuous* [i.e., relative to the natural profinite group topology defined in (O2) on $\mathrm{Aut}(\mathcal{G}[c])$].

It is immediate that (c^{new}) implies (c). Moreover, we observe in passing that:

(O3) In fact, since H is *topologically finitely generated* [cf. Definition 5.1, (i), (a)], it holds [cf. [NS], Theorem 1.1] that *every finite index subgroup of H is open in H*. Thus, the conditions (c) and (c^{new}) in fact *hold automatically*.

The *new version* of the condition (d) of Definition 5.1, (i), that we wish to consider is the following:

(d^{new}) There is a *finite* set C^* of *components* [i.e., vertices and edges] of $\mathbb{G}$ such that for every component c of $\mathbb{G}$, there exists a $c^* \in C^*$ and an *isomorphism* of semi-graphs of anabelioids $\mathcal{G}[c] \xrightarrow{\sim} \mathcal{G}[c^*]$ that is *compatible* with the action of H on both sides.

It is immediate that (d^{new}) implies (d). The reason that, in the context of the proof of Proposition 5.2, (i), it is necessary to consider the *stronger conditions* (c^{new}) and (d^{new}) is as follows. It suffices to show that, given a *connected finite étale covering* $\mathcal{G}' \to \mathcal{G}$, after possibly replacing H by an open subgroup of H, the action of H on $\mathcal{G}$ *lifts* to an action on $\mathcal{G}'$ that satisfies the conditions of Definition 5.1, (i). Such a lifting of the action of H on $\mathcal{G}$ to an action on the portion of $\mathcal{G}'$ that lies over the *vertices* of $\mathbb{G}$ follows in a straightforward manner from the *original* conditions (a), (b), (c), and (d). On the other hand, in order to conclude that such a lifting is [after possibly replacing H by an open subgroup of H] *compatible with the gluing conditions* arising from the structure of $\mathcal{G}'$ over the *edges* of $\mathbb{G}$, it is necessary to assume further that the *"component-wise versions (*c^{new}*), (*d^{new}*)"* of the original *"vertex-wise conditions (c), (d)"* hold. This issue is closely related to the issue discussed in (9.) above.

(14.) In Definition 2.4, (iii), the phrase "underlying graph" should read "underlying semi-graph" (2 instances).

(15.) In the first sentence of the fourth paragraph of the discussion entitled "Curves" in §0, the notation "$\overline{\mathcal{D}}_{g,r} \subseteq \overline{\mathcal{M}}_{g,r}$" should read "$\overline{\mathcal{D}}_{g,r} \subseteq \overline{\mathcal{C}}_{g,r}$".

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